II Surfaces (Kirby pictures, classification, and diffeomorphisms)

we will use handlebodies to classify closed surfaces

First we have

exercise:

1) if Σ a surface and D_1, D_2 are 2 disks in Σ then

$$\overline{\Sigma - D_1} \cong \overline{\Sigma - D_2}$$

so denote $\overline{\Sigma - D_1}$ by $\widehat{\Sigma}$

2) given Σ_1, Σ_2 the connected sum is

$$\Sigma_1 \# \Sigma_2 = \widehat{\Sigma}_1 \cup_f \widehat{\Sigma}_2 \quad \text{where } f: \partial\widehat{\Sigma}_1 \to \partial\widehat{\Sigma}_2$$

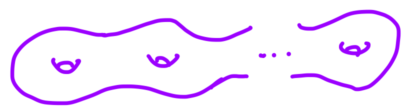
is an or$^{\underline{n}}$ rev. diffeo ← unique upto isotopy

show this is well-defined.

examples:

1) S^2  call Σ_0

2) $T^2 = S^1 \times S^1$  call Σ_1

3) $\Sigma_n = \Sigma_{n-1} \# \Sigma_1$ 

projective plane → 4) $P^2 = S^2/\sim$ call N_1

↑ ident antipodes

5) $N_n = N_{n-1} \# N_1$

Th$^{\underline{m}}$:

$$\boxed{\text{any closed connected surface } \Sigma \text{ is diffeomorphic to } \Sigma_n \text{ or } N_n \text{ for some } n}$$

we prove this by showing Σ has a handlebody decomp. that agrees with one for Σ_n or N_n

how to "see" handle decomps? Kirby pictures!

that is, we will represent any closed

connected surface with a diagram in $\mathbb{R}^1$

later we will see a 3-manifold can be represented by a diagram in $\mathbb{R}^2$

and a 4-manifold by a diagram in $\mathbb{R}^3$

<u>note</u>: for surfaces F_0, F_1, we have $F_0 \cong F_1 \Leftrightarrow \widehat{F}_0 \cong \widehat{F}_1$

so given a connected surface F we can find a handle decomposition with one 0-handle and one 2-handle

and we can understand F's diffeomorphism type by considering
$F - (2\text{-handle}) = (0\text{-handle}) \cup k(1\text{-handles})$

to understand this, just need to know how 1-handles are attached to $\partial(0\text{-handle}) = S^1$

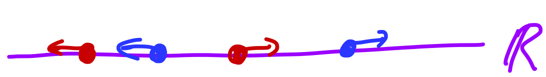
i.e keep track of framed S^0's in S^1

called "pointed matched circle" in bordered HF

we can assume S^0's miss some point in S^1 so think of S^0's as in $S^1 - \{pt\} \cong \mathbb{R}$

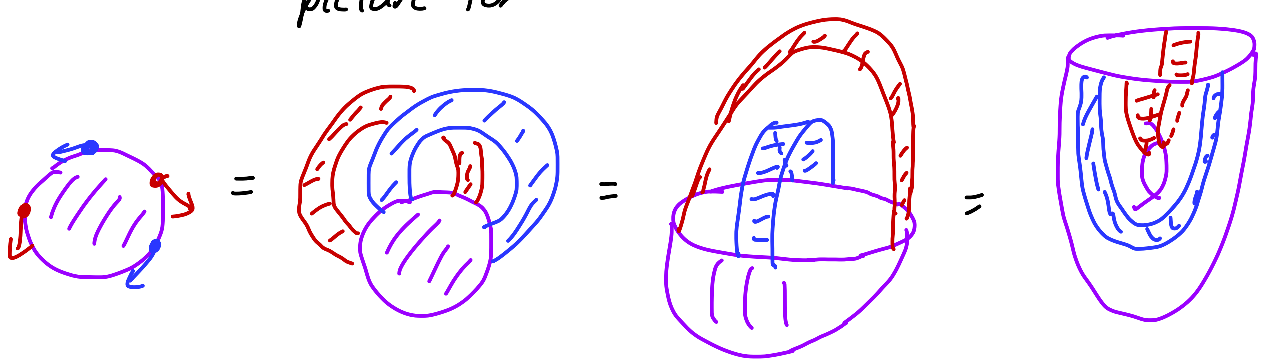
reduced studying 2-mfds to pictures in $\mathbb{R}^1$! call these "Kirby pictures"

<u>example</u>: 1)

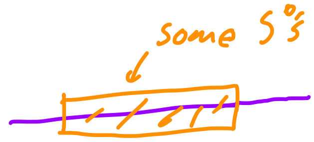


picture for

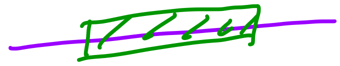
$\hat{T}^2 = \overline{T^2 - D^2}$

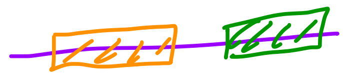


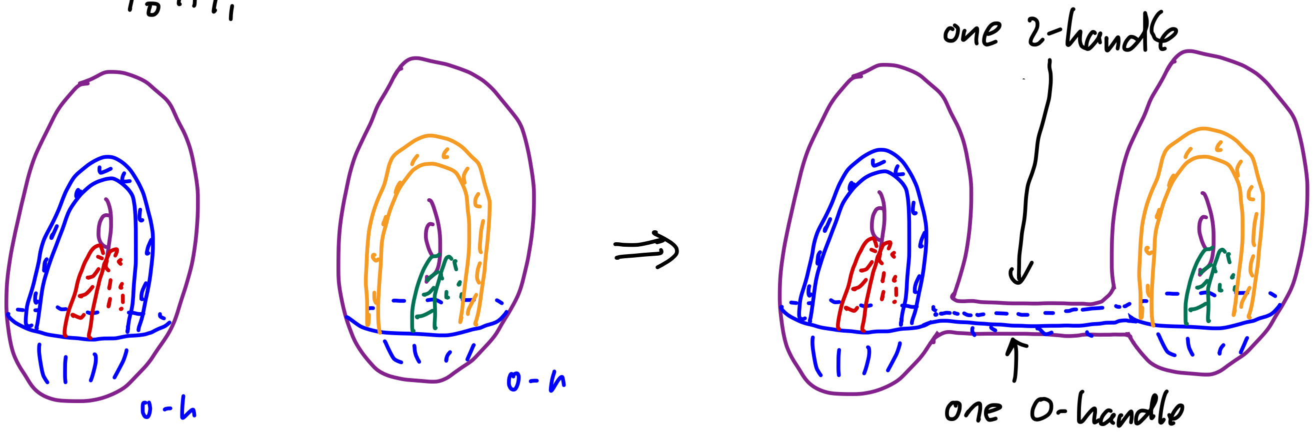
so this is a picture of T^2

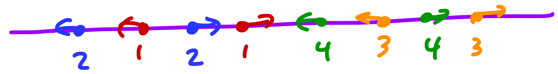
2) if F_0 has picture 

and

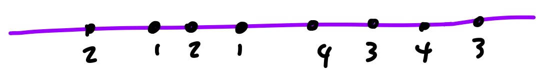
F_1 " "  then

$F_0 \# F_1$ " " 

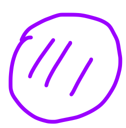


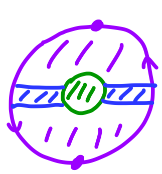
e.g Σ_2 has picture 

or



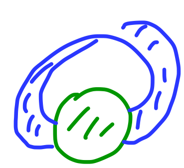
with convention, if no arrows then they point "out"

3) $P^2 =$  $\cong$  with pt on ∂ identified with "antipode"

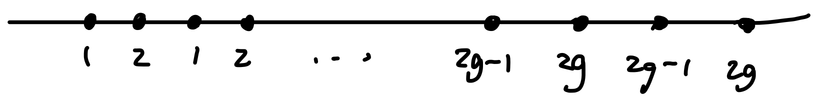


note union of purple is D^2 glued to rest along S^1, i.e. a 2-handle

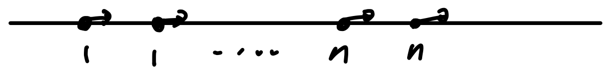
remove 2-handle to get

 $=$ 

$=$ 

so Σ_n is 

where $\Sigma_0 = S^2$ is

and N_n is 

so we now have pictures of all Σ_n and N_n !

We now show any F has a picture that agrees with one of these

- If the picture for F has no 1-handles, then $F \cong S^2$
- now suppose F has $k>0$ 1-handles

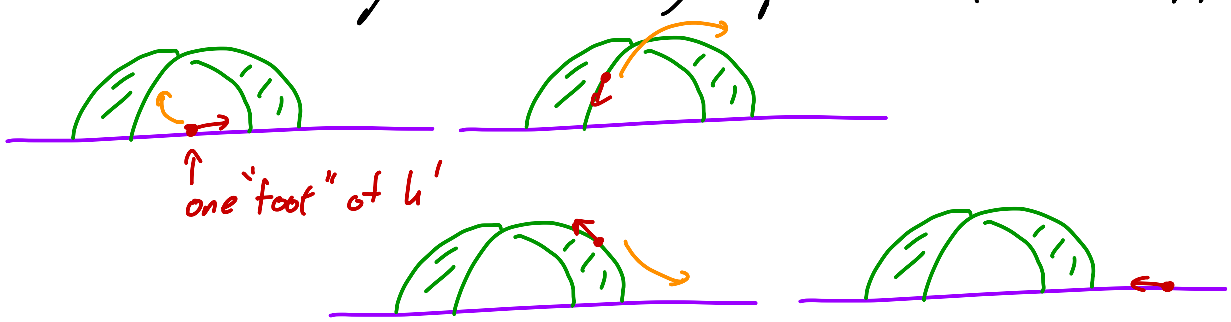
consider one of them: h_0^1 that is attached first

<u>Case 1:</u> h_0^1 not oriented

any handle h^1 attached after h_0^1 can be "slid" over h_0^1

that is recall we get the same manifold if

we change the attaching map of h^1 by an isotopy

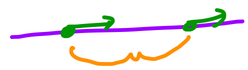


so a "handle slide" changes the picture by

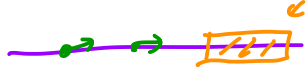
exercise: Show handle slide rule is

a) can push a point into tip/tail of another handle h_0^1 and it comes out the tip/tail of the other "foot" of h_0^1

b) if h_0^1 is oriented the arrow of the slid point stays same, otherwise it flips

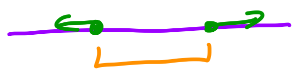
so notice any "foot" in  can be slid out of I (say to right)

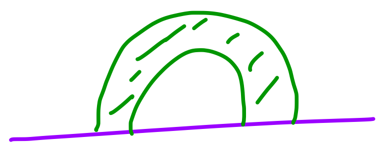
and any point to left of I can be slid to right of I by 2 handle slides

so $F =$ 

$\therefore F \cong P^2 \# F'$ F' has fewer 1-handles

Case 2: h_0^1 oriented





note: $\partial((0\text{-}h) \cup h_0^1) = S^1 \cup S^1$ not connected

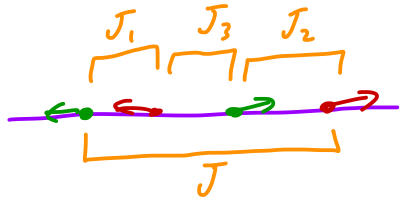
but $\partial F^0 = S^1$ connected

so must be h_1^1 with one foot in I and one foot out

if h_1^1 non oriented, then can slide to get

so now in Case 1 and $F = P^2 \# F'$ ← fewer 1-handles

if h_1^1 oriented, then consider interval J



note: any foot in J_1 can be slid over h_1' to get out of J

" " J_2 " " h_0' " "

" " J_3 " " h_1' to get to J_2

then over h_0' to get out of J

similarly any foot to left of J can be slid to right of J

$\therefore$ as in case 1 $F = T^2 \# F'$ with F' having fewer 1-handles

so by induction on k $F = (\#_k T^2) \# (\#_\ell P^2)$

exercise: $T^2 \# P^2 \cong P^2 \# P^2 \# P^2$

so we are done with proof of theorem!

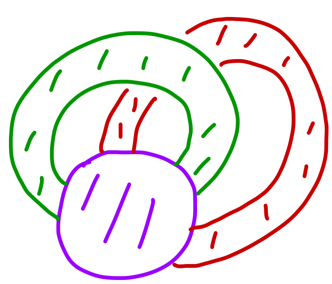
Subtle Point: Given a handlebody F and F' obtained from F by isotoping the attaching maps,

Is $F = F'$?

NO! but they are diffeomorphic!

there is a difference between "is the same as" and "diffeomorphic to"

example:



isotope attaching region of red as follows

so it comes back to where it started

we get a family of surfaces F_t $t \in [0,1]$

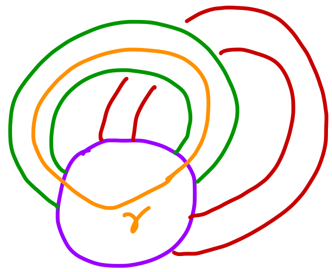
with $F = F_0 = F_1$ and diffeomorphisms

$\phi_t : F_0 \to F_t$

really equal!

exercise: so $f_1 : F \to F$ is a diffeomorphism of F!

1) Show f_1 is isotopic to a "Dehn twist" about γ



2) Show any diffeomorphism of an oriented surface is obtained by handle slides (is it true for non orientable?)

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